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ABSTRACT

This paper discusses methods of sketching various types of algebraic functions from an analysis of the portions of the plane where the curve will be found and where it will not be found. The discussion is limited to rational functions. Methods and techniques presented are applicable to the secondary mathematics curriculum from algebra through calculus. The authors also offer a list of advantages which can be derived from using the techniques described. (FL)

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Sign Lines, Asymptotes, And Tangent Carriers ---
An Introduction to Curve Sketching

by

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Three cheers for Maurice Nadler's, "The Demise of Analytic Geometry."[1] It is encouraging to know that there are those who yearn for the "good old days" when the stuff of analytic geometry was an important part of the mathematics curriculum. Especially when, as Nadler so aptly states, "...calculus textbooks sometimes force the exclusive use of calculus techniques when the material may also reside in the domain of analytic geometry."[2]

A review of a random sample of current calculus texts revealed that all have sections or units on curve sketching. As Nadler indicates, "Many problems of curve sketching are amenable to the ingenuity of the analytic geometry approach."[3] In support of this position, we offer the following ideas and techniques on curve sketching as further evidence that "analytic geometry and calculus can live peacefully together side by side."[4]

Our aim is to develop methods of sketching various types of algebraic functions from an analysis of those portions of the (x,y) plane where the curve will be found, and, just as importantly, where it will not be found. We limit our discussion to rational functions of the form $y=f(x)=\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomial functions in factored form with no common factors.

Sign Lines

Consider the function $y = \frac{x+1}{x-3}$. (We recognize, of course, that this second degree function in x and y is a simple rectangular hyperbola and may be easily sketched from this viewpoint. We have chosen it, however, because it affords an opportunity to illustrate basic ideas developed here.) The zeros of the numerator and denominator, respectively, are $x = -1$ and $x = 3$. In Figure 1, we draw a dotted line, parallel to the y-axis, through each of these zeros.

These lines ($x = -1$ and $x = 3$) separate the (x,y) plane into three regions. Regions I is to the right of $x = 3$; region II is between $x = -1$ and $x = 3$; and, region III is to the left of $x = -1$. We wish to examine the behavior of the function in each of these regions.

We see that for $x > 3$, $y > 0$. That is, in region I, y is positive and the graph <u>cannot</u> fall below the x-axis. For $-1 < x < 3$, $y < 0$. That is, in region II, y is negative and the graph <u>cannot</u> fall above the x-axis. For $x < -1$, $y > 0$. That is, in region III, y is positive and the graph <u>cannot</u> fall below the x-axis.

Figure 2 illustrates these ideas. The shaded portion of each region represents that part of the (x,y) plane where the graph cannot fall. The unshaded portion of each region represents that part of the (x,y) plane where the graph can fall.

It is interesting to note that as we move right to left (or left to right) across the lines $x = 3$ and $x = -1$, in Figure 2, the shading alternates with respect to the x-axis. That is, on one side of the line the shading is below the x-axis and on the other side it is above the x-axis. This is, in fact, true because the behavior of y is changing sign from positive to negative (or vice-versa) as we move from any region to the adjoining one.

We say that $x = 3$ and $x = -1$ are vertical sign lines for the function. Vertical sign lines are indicated in sketches by dotted vertical lines labeled vsl.

Definition I: Given $y = f(x)$. If $y = f(x)$ changes sign at $x = a$, then $x = a$ is a vertical sign line (vsl) for $f(x)$.

Three important characteristics of a vertical sign line will be of great value in sketching rational functions. First, a vertical sign line may be determined by finding the zero of an odd powered factor of either the numerator or denominator of $f(x)$. Second, the shaded portions of the (x,y) plane alternate above and below the x-axis as we move across a vertical sign line.

Third, if a vertical sign line is established by a zero of the numerator of $f(x)$, then the intersection of the vertical sign line and the x-axis determines an x-intercept of $f(x)$.

Vertical Asymptotes

Rewriting $y = \frac{x + 1}{x - 3}$ with x in terms of y and then writing x as a mixed expression yields $x = 3 + \frac{2}{y - 1}$. An analysis of the fractional term in the mixed expression reveals that for $y > 1$, $x > 3$ and x approaches 3 from the right as the vertical distance from $y = 1$ increases. Also, for $y < 1$, $x < 3$ and x approaches 3 from the left as the vertical distance from $y = 1$ increases. That is, as $|y| \to \infty$, $x \to 3$. (We observe that we could have obtained these results by noting that $x = 3$ is not in the domain of the function and that y approaches positive infinity when x approaches 3 from the right and y approaches negative infinity when x approaches 3 from the left.)

We say that $x = 3$ is a vertical asymptote for the function. Vertical asymptotes are indicated in sketches by dotted vertical lines labeled <u>va</u>.

Since $x = 3$ is also a vertical sign line for the function, we have already established the behavior of the graph in the regions of the (x,y) plane determined by the vertical asymptote $x = 3$.

Definition II: Given $y = f(x)$. If $|y| \to \infty$ as $x \to a$, then $x = a$ is a vertical asymptote (va) for $f(x)$.

Three important characteristics of a vertical asymptote will be of great value in sketching rational functions. First, a vertical asymptote may be determined by finding a zero of the denominator (and only the denominator) of a function. Second, if a vertical asymptote is a vertical sign line ($x = a$ is a zero of an odd powered factor of the denominator), then the shaded portions of the (x,y) plane alternate above and below the x-axis as we move across the vertical asymptote. Third, the graph of a function will approach (but never touch) one or both extremities of a vertical asymptote.

Horizontal, Oblique, and Curvilinear Asymptotes

Writing $y = \dfrac{x + 1}{x - 3}$ as a mixed expression yields $y = 1 + \dfrac{4}{x - 3}$. In Figure 3 we draw a dotted line parallel to the x-axis through 1 (the integral expression in $1 + \dfrac{4}{x - 3}$) and a dotted line parallel to the y-axis through 3 (the zero of the denominator of the fractional term in $1 + \dfrac{4}{x - 3}$).

An analysis of the fractional term in the mixed expression reveals that for $x > 3$, $y > 1$ and y approaches 1 from above as the horizontal distance from $x = 3$ increases. Also, for $x < 3$, $y < 1$ and y approaches 1 from below as the horizontal distance from $x = 3$ increases. That is, as $|x| \to \infty$, $y \to 1$.

We say that $y = 1$ is a horizontal asymptote for the function. Horizontal asymptotes are indicated in sketches by dotted horizontal lines labeled ha.

We can analyze the behavior of the function with respect to the horizontal asymptote. As noted above, for $x > 3$, $y > 1$. That is, for $x > 3$ the graph of the function cannot fall below the line $y = 1$. Also, for $x < 3$, $y < 1$. That is, for $x < 3$ the graph cannot fall above the line $y = 1$. This analysis sheds additional information on regions of the (x,y) plane where the graph of the function can or cannot fall.

Figure 4 illustrates the behavior of y with respect to the horizontal asymptote $y = 1$. The shaded portion of each region represents that part of the (x,y) plane where the graph cannot fall. The unshaded portion of each region represents that part of the (x,y) plane where the graph can fall.

It is interesting to observe the similarity between the vertical sign line notion and the behavior of the function with respect to the horizontal asymptote. If $x = a$ is a vertical sign line, $y = f(x)$ changes sign from positive to negative (or negative to positive) at $x = a$. Our discussion for the horizontal asymptote reveals that $y = f(x)$ changes from greater than 1 to less than 1 (or less than 1 to greater than 1) at $x = 3$. We can think of $x = 3$ as being a sign line with respect to the horizontal asymptote, $y - 1 = 0$. That is, $y - 1$ changes sign at $x = 3$. Thus, $x = 3$ is a vertical sign line with respect to $y - 1 = 0$; and, vice-versa, $y = 1$ is a sign line with respect to $x - 3 = 0$.

The line $x = 3$ separates the (x,y) plane into two regions. One region is the right of $x = 3$ (and above or below $y = 1$) and the other region is to the left of $x = 3$ (and above or below $y = 1$). As we move right to left (or left to right) across the line $x = 3$ in Figure 4 the shading alternates with respect to the line $y = 1$.

We have obtained information on the behavior of the graph of the function in our analysis of the vertical sign line (Figure 2) an in our analysis of the horizontal asymptote (Figure 4). 5 We now combine this information in one sketch in Figure 5. As before, the shaded portions of the (x,y) plane indicate where the graph cannot fall. Horizontal (right-left) shading represents those portions of the plane where the graph cannot fall as revealed by an analysis of the vertical sign lines with respect to $y = 0$.

Vertical (up-down) shading represents those additional portions of the plane where the graph cannot fall as revealed by an analysis of the sign line with respect to the horizontal asymptote.

Before defining the horizontal asymptote, we note that the integral expression in $y = 1 + \frac{4}{x - 3}$ is a constant and the asymptote, therefore, is of the form $y = b$ (a constant). This will not always be the case. Depending upon the rational function under consideration, the integral expression may be a higher powered function of x rather than a constant. We do not present a discussion of such asymptotes at this time but include them in the definition and present some illustrative examples later in the paper.

Definition III: Given $y = f(x)$. If $y \to g(x)$ as $|x| \to \infty$, then $y = g(x)$ is an asymptote for $f(x)$. We call $g(x)$:

a) a horizontal asymptote (ha) if $g(x) = b$ (a constant),

b) an oblique asymptote (oa) if $g(x) = mx + b$ ($m \neq 0$) and,

c) a curvilinear asymptote (ca) if the graph of $y = g(x)$ is not a straight line.

Four important characteristics of a horizontal asymptote (and, similarly, oblique and curvilinear asymptotes) will be of great value in sketching rational functions. First, a horizontal asymptote may be determined by writing f(x) as a mixed expression. If the integral expression(in the mixed expression) is a constant b, then y = b will be the horizontal asymptote. Second, the graph of a function will always approach (but never touch) the extremities of the horizontal asymptote. Third, an analysis of the fractional term in the mixed expression sheds additional light on regions of the (x,y) plane where the graph can or cannot fall. Zeros of odd powered factors of the fraction establish sign lines with respect to the horizontal asymptote, y = b. The shaded portions of the (x,y) plane alternate above and below the asymptote as we move across these sign lines. Fourth, if a graph crosses its horizontal asymptote, the intersection of the graph and the asymptote is easily determined by finding a zero of the numerator of the fractional part of the mixed expression.

Intercepts

A serious discussion of intercepts seems unnecessary. This topic is covered well in mathematics literature.

Definition IV: Given $y = f(x)$. If $y = 0$ when $x = a$, then $x = a$ is an x-intercept of $f(x)$.

Definition V: Given $y = f(x)$. If $y = b$ when $x = 0$, then $y = b$ is a y-intercept of $f(x)$.

As we noted earlier the intersection of a vertical sign line, that is not also a vertical asymptote, and the x-axis determines an x-intercept for $f(x)$.

For the function $y = \frac{x + 1}{x - 3}$, $-\frac{1}{3}$ is the y-intercept and, -1 is the x-intercept. Intercepts are indicated in sketches by dots on the appropriate axes.

The Sketch

Now, using Figure 5 and our knowledge of sign lines, asymptotes, and intercepts, we are able to easily sketch $y = \frac{x+1}{x-3}$. It is not our intention to produce exact graphs of functions. We are concerned only with presenting freehand sketches that accurately reflect the behavior of the function.

Figure 6 presents the completed sketch for $y = \frac{x+1}{x-3}$. In actual practice the sketch is made region by region. For example, in region I of Figure 6 (to the right of $x = 3$) we know that the curve must approach ~~from~~ above the far right extremity of the horizontal asymptote. Also, the curve must approach the upper extremity of the vertical asymptote as we approach $x = 3$ from the right. There is but one way that these conditions may be satisfied and the result is the branch of the curve drawn in region I.

Similarly, in region II we know that the curve must pass through the intercepts and approach the lower extremity of the vertical asymptote as we approach $x = 3$ from the left. Again, there is but one way that these conditions may be satisfied and the result is the branch of the curve drawn in region II. In region III the curve must approach from below the far left extremity of the horizontal asymptote. This condition results in the branch of the curve drawn in region III.

Before expanding upon the techniques above, we present another example to focus attention on the simplicity of sketching rational functions by means of sign lines, asymptotes, and intercepts. Consider the function $y = \frac{(x+1)(x-1)}{(x+2)(x-3)}$. The step-by-step procedure outlined below leads quickly to the sketch in Figure 7.

1. Draw the x and y axis.
2. Identify the vertical sign lines by finding zeros of odd powered factors of the numerator and denominator of the function. In this case, $x=1$, $x=-1$, $x=-2$, and $x=3$ are the vertical sign lines.

3. Locate and label the vertical sign lines by drawing dotted lines through the zeros and writing vsl beneath the lines.

4. Shade the regions of the (x,y) plane where the graph cannot fall. Testing one region of the plane, $x > 3 \Rightarrow y > 0$, we see that to the right of $x = 3$ the portion below the x-axis should be shaded. Remembering that on either side of a vertical sign line the shaded regions alternate above and below the x-axis, it is a simple matter to alternate our shading in the remaining regions of the plane.

5. Identify the vertical asymptotes by finding zeros of the denominator of the function. In this case $x=-2$ and $x=3$ are vertical asymptotes.

6. Label the vertical asymptotes by writing va beneath the lines $x=-2$ and $x=3$.

7. Identify other asymptotes by writing the function as a mixed expression. $y = \frac{(x+1)(x-1)}{(x+2)(x-3)} = 1 + \frac{(x+5)}{(x+2)(x-3)}$. In this case $y=1$ is the horizontal asymptote since 1 is the integral expression in the mixed expression. (We observe that still another way to identify the horizontal asymptote is to examine the quotient of the highest degreed terms in the numerator and denominator of $f(x)$.)

8. Locate and label the horizontal asymptote by drawing the dotted line $y=1$ and writing ha beside it.

9. Shade the additional regions of the (x,y) plane where the graph cannot fall. Finding zeros of the numerator and denominator of the fractional term in the mixed expression $y = 1 + \frac{x+5}{(x+2)(x-3)}$ reveals that $x=-5$, $x=-2$, and $x=3$ are sign lines with respect to $y=1$. Testing one region of the plane $x > 3 \Rightarrow y > 1$, we see that to the right of $x=3$ the portion below the line $y=1$ should be shaded. Remembering that on either side of a sign line the shaded regions alternate above and below the horizontal asymptote, it is a simple matter to alternate our shading in the remaining regions of the plane. A note of caution: Be careful not to confuse the sign lines $x=-5$, $x=-2$, and $x=3$ with the vertical sign lines for $f(x)$. The shading with respect to the asymptote alternates only as we cross $x=-5$, $x=-2$, and $x=3$. We suggest that these vertical sign lines be labeled vsl at the top of the dotted lines. This will help avoid confusing the two sets of sign lines and vertical sign lines.

10. Identify and locate points of intersection for the graph and the horizontal asymptote by finding zeros of the

numerator in the fractional part of the mixed expression $y=1+\frac{x+5}{(x+2)(x-3)}$. In this case $x=-5$ is the only such zero. Therefore, $(-5,1)$ is the only point of intersection. We locate $(-5,1)$ by placing a dot at the appropriate point in the plane.

11. Identify and locate the x-and-y-intercepts. In this case the y-intercept is $\frac{1}{6}$ and the x-intercepts are -1 and 1. We locate the intercepts by placing dots at the appropriate place on the axes.

12. Identify and locate relative high and low points of $f(x)$ by solving for x as a function of y and observing that the discriminant cannot be negative for real values of x. That is, $y^2 - 4(y-1)(1-6y) \geq 0$. The high and low points, respectively, are $\left(-5+2\sqrt{6}, \frac{14-4\sqrt{6}}{25}\right)$ and $\left(-5-2\sqrt{6}, \frac{14+4\sqrt{6}}{25}\right)$. We locate these points by placing appropriate dots in the plane.

13. Sketch the curve region by region.

Tangent Carriers

To complete our methods and techniques for sketching rational functions, consider, first, the function $y=(x-3)^2$. We see that

$x=3$ is a zero of the numerator and, as discussed earlier, drawing a dotted line through $x=3$, parallel to the y-axis, would separate the (x,y) plane into two regions: one to the right of $x=3$ and the other to the left of $x=3$.

For $x > 3$, $y > 0$ and for $x < 3$, $y > 0$. That is, for $x > 3$ the graph of the function cannot fall below the x-axis and, similarly, for $x < 3$ the graph cannot fall below the x-axis. Figure 8 illustrates this analysis.

As one might expect, when we move right to left (or left to right) across the line $x=3$ the shading in the (x,y) plane does not alternate above and below the x-axis. Instead, the shading remains the same -- above on both sides or below on both sides of $x=3$. This is the case because $y= (x-3)^2$ is a parabola tangent to the x-axis at $x=3$.

Here we say that $x=3$ is a tangent carrier on $y=0$ for the function. Tangent carriers are indicated in sketches by dotted lines labeled tc.

Definition VI: Given $y=f(x)$. If $f(a)=0$ and $f(x)$ does not change sign at $x=a$, then $x=a$ is a tangent carrier (tc) on $y=0$ for $f(x)$.[6]

Four important characteristics of a tangent carrier will be of great value in sketching rational functions. First, a tangent carrier may be determined by finding a zero of an even powered factor of the numerator (and only the numerator) of the function. Second, the intersection of a tangent carrier and the x-axis determines an x-intercept of the function. Third, with respect to the x-axis, the shaded portions of the (x,y) plane remain the same on both sides of a tangent carrier. Fourth, if f(x) can be written as a mixed expression, additional information may be obtainined on regions of the (x,y) plane where the graph can or cannot fall (as explained earlier). A zero of an even powered factor of the numerator (and only the numerator) of the fractional part of the mixed expression establishes a tangent carrier with respect to the horizontal asymptote. The shaded portions of the (x,y) plane remain the same on both sides of a line which is a tangent carrier with respect to the horizontal asymptote.

In Figure 9 we present a completed sketch for $y = (x-3)^2$.

Asymptotic Tangent Carriers

Consider the function $y = \frac{1}{(x-3)^2}$. We see that x=3 is a zero of the denominator and, as discussed earlier, drawing a dotted line through x=3, parallel to the y-axis, would separate the (x,y) plane into two regions.

For $x > 3$, $y > 0$ and for $x < 3$, $y > 0$. That is, for $x > 3$

the graph of the function cannot fall below the x-axis and, similarly, for $x < 3$ the graph cannot fall below the x-axis.

This analysis is identical to that given for the tangent carrier. However, since $x=3$ is a zero of the denominator it is also a vertical asymptote. Consequently, the graph of the function does not pass through the point (3,0). That is, the curve is not tangent to $y=0$ at $x=3$ and $x=3$ is, therefore, not a tangent carrier. Here we say that $x=3$ is an asymptotic tangent carrier. Asymptotic tangent carriers are indicated in sketches by dotted vertical lines labeled atc.

Definition VII: Given $y=f(x)$. If $f(a)$ is undefined and $f(x)$ does not change sign at $x=a$, then $x=a$ is an asymptotic tangent carrier (atc) for $f(x)$.

Three important characteristics of an asymptotic tangent carrier will be of great value in sketching rational functions. First, an asymptotic tangent carrier may be determined by finding a zero of an even powered factor of the denominator (and only the denominator) of the function. Second, the graph of a function will always approach from both sides (but never touch) one extremity of an asymptotic tangent carrier. Third, the shaded regions of the (x,y) plane remain the same on both sides of an asymptotic tangent carrier.

Figure 10 presents a completed sketch for $y = \frac{1}{(x-3)^2}$. Note that y=0 is a horizontal asymptote for the function since $y \to 0$ as $|x| \to \infty$.

Summary

Prior to illustrating the flexibility of the above techniques for graphing various rational functions, we present a summary of the ideas and methods discussed. These points should enable readers to synthesize the notions of sign lines, asymptotes, tangent carriers, and asymptotic tangent carriers.

Let $y = f(x) = \frac{N(x)}{D(x)} = g(x) + \frac{R(x)}{D(x)}$, where N(x) and D(x) are polynomial functions of x with no common factors, g(x) is a polynomial function of x, and R(x) and D(x) are polynomial functions of x with no common factors. Let n be a positive integer in each of the following cases:

1. If $(x-a)^{2n-1}$ is a factor of N(x), then x=a is a vertical sign line and an x-intercept of f(x).
2. If $(x-a)^{2n-1}$ is a factor of D(x), then x=a is a vertical sign line and a vertical asymptote of f(x).
3. If $(x-a)^{2n}$ is a factor of N(x), then x=a is a tangent carrier on y=0 and an x-intercept of f(x),

4. If $(x-a)^{2n}$ is a factor of $D(x)$, then $x=a$ is an asymptotic tangent carrier that is not a sign line for $f(x)$.
5. If $y \to g(x)$ as $|x| \to \infty$, $y=g(x)$ is a horizontal, oblique, or curvilinear asymptote of $f(x)$.
6. Similar comments may be made concerning sign lines and tangent carriers (obtained from $\frac{R(x)}{D(x)}$) with respect to $y=g(x)$.
7. If it is possible to express the function for x in terms of y additional information about $y=f(x)$ may be obtained. Using the quadratic formula, restrictions on the range may be obtained by noting that the discriminant cannot be negative. Also, an analysis of non-vertical sign lines (see footnote 5) may shed additional light on regions of the (x,y) plane where the graph can or cannot fall.

Some Interesting Sketches

The limitations of space prevent an extensive explanation of the application of the above techniques to the sketching of rational functions. However, for the reader who may wish to use these techniques, we include below completed sketches of some interesting rational functions used in our classes.

Figure 11 illustrates a function that has a tangent carrier and an asymptotic tangent carrier.

Figure 12 illustrates a function for which the relative high and low points may be found by using the quadratic formula. The relative high and low points are, respectively, (1,1) and $(\frac{-5}{3},9)$.

Figure 13 illustrates a function (hyperbola) that has an oblique asymptote, $y=x+5$. The relative high and low points of the function are, respectively, (-4,0) and (-2,4).

Figure 14 illustrates a function that has a curvilinear asymptote, $y=x^2$.

Figure 15 illustrates a function that crosses its horizontal asymptote, at (2,1). The relative high and low points of the function are, respectively, (0,0) and $(4,\frac{8}{9})$.

Figure 16 illustrates a function that has a tangent carrier on a line that is not an asymptote or the x-axis, the line $y=2$, and which has point symmetry, about (0,1). The following analysis may be helpful in recognizing that $x=1$ is a tangent carrier on $y=2$.
$$y=1+\frac{2x}{x^2+1} \Leftrightarrow y=2+\frac{2x}{x^2+1}-1 \Leftrightarrow y=2-\frac{(x-1)^2}{x^2+1}.$$

Figures 17 and 18 illustrate, visually, that the techniques discussed in this paper have further applications and may be applied in the sketching of various multi-valued functions.

Solving Inequalities By Curve Sketching

There seems to have been a magic for us in the October, 1969 Mathematics Teacher. Nadler's article spurred us on to put down on paper the ideas and techniques that we have been using in the classroom. Admittedly, another article, Henry Frandsen's "The Last Word on Solving Inequalities,"[7] in the same issue of the Mathematics Teacher, had a similar effect.

Frandsen's hypothesis, "The final technique (referring to methods for problem solving) to be emphasized should be the most efficient one that can be made available to the students at their particular level."[8] is an excellent one. He presents an interesting technique for solving inequalities, and concludes that, "The simplicity of this method makes it a candidate for 'the last word' in any unit on solving inequalities."[9]

We quite agree. However, it is our belief that a technique is truly valuable when, with efficiency, it produces maximum understanding and possesses numerous additional applications. If we want students to think about where functions increase, decrease, or reach relative maxima and minima, then by all means let us show them, visually, what it is we are talking about.

For this reason we advocate the use of sign lines, asymptotes, and tangent carriers in the solution of inequalities. As illustated by the example below, these techniques enable the student who solves an inequality to see the behavior of the function. Additionally, these techniques are very useable in the sense that whenever sketches of rational functions are desired, the methods discussed in this paper will be of value.

As our example of the use of curve sketching in the solution of inequalities, we use the problem discussed by Frandsen in his article. That is, solve $\frac{(x-1)(x+2)(x+3)}{(2x+1)(x)(x-2)} \leq 0$.

To obtain the solution of this inequality we graph the function $y = \frac{(x-1)(x+2)(x+3)}{(2x+1)(x)(x-2)}$. Certain steps in our procedure have been eliminated in making the sketch because they provide information not absolutely necessary for determining the behavior of the function. If the reader refers back to the section of this paper headed "The Sketch," it will be obvious which steps have been eliminated in arriving at the sketch presented in Figure 19.

From the sketch in Figure 19 we may readily determine the values of x for which the function is negative or zero. We need only look for those places where the graph of the function is below or on the x-axis. These values of x are: $-3 \leq x \leq -2$, $-\frac{1}{2} < x < 0$, and $1 \leq x < 2$; and in fact, are the solutions of the inequality $\frac{(x-1)(x+2)(x+3)}{(2x+1)(x)(x-2)} \leq 0$. Of course, it is actually not even necessary to graph the function to solve the inequality since the shaded regions alone will indicate where it is positive or negative.

This method can be used to find the solution set of any inequality involving $\frac{N(x)}{D(x)}$, where N(x) and D(x) are algebraic functions in factored form

Conclusion

We would like to acknowledge here the work of Dr. Howard Fehr, Teachers' College, Columbia University. Many years ago one of the authors was a student in a graduate mathematics course where Dr. Fehr introduced the ideas of sign lines, asymptotes, and tangent carriers. Our work is based upon the ideas presented by Dr. Fehr. We have developed the methods and techniques in such a way as to make them applicable in the secondary mathematics curriculum at every level from algebra through calculus.

In a 1951 text for college seniors and graduate students, Secondary Mathematics: A Functional Approach for Teachers,[10] Dr. Fehr included a brief chapter on "Curve Tracing." He stated, "The method of curve tracing presented here is not intended for the regular high school course..."[11] and concluded the chapter by posing the question, "What values has this method of curve tracing in the study of differential calculus in college?"[12]

We have found these techniques to be very valuable for calculus students. Often, they enhance the insight provided by the calculus. However, an even greater value for these techniques can be found in their practical and useful application in the secondary classroom. They have been tested by actual classroom use, at every level from algebra through calculus, with excellent results. We offer for the reader's consideration some of the advantages we feel can be dervied from the use of sign lines, asymptotes, and tangent carriers in the secondary curriculum.

1. <u>They require a mastery of fundamental algebraic skills.</u> Operations such as factoring, simplifying algebraic fractions, working with mixed rational expressions, solving linear and quadratic equations and inequalities, and use of the quadratic formula are fundamental in the use of the techniques.
2. <u>They require some thought and analysis.</u> The processes are not entirely automatic, and the student is able to develop insight and analytical skills through practice and experience.
3. <u>They bring into focus some practical uses for basic algebraic skills.</u> Many computational and manipulative skills, previously acquired by students, often appear

to be of dubious value. The methods of curve sketching afford an opportunity to demonstrate practical applications for such skills.

4. They introduce ideas and skills often not studied until the calculus. When students are able to grasp advanced concepts, the advantage of introducing such material prior to the study of calculus is obvious. Among the ideas that these techniqeus bring to the student's attention are: increasing and decreasing functions, maxima and minima points, domain, range, and limits.
5. They may be presented as a new unit of study or they may serve as a review of previously studied material. The techniques may be used an an unusual and unique approach to reviewing the fundamental skills of algebra and the student's knowledge of lines and conic sections; or, the material can be used to initiate the analytic study of functions.
6. They serve as an unusual motivational tool for classroom use. Students at every level find the material different, interesting, and entertaining.
7. They enable students to solve many curve sketching problems such as those found on the advanced placement test.

While this is certainly not an end in itself, it does indicate that perhaps this aspect of the study of calculus can be introduced earlier in the curriculum.

Footnotes

[1] Maurice Nadler, "The Demise of Analytic Geometry," The Mathematics Teacher, LXII (october, 1969), 447-452

[2] ibid, p. 450

[3] ibid

[4] ibid

[5] There are additional portions of the (x,y) plane where it can be determined that the graph of the function does or does not fall. Rewriting $y=\frac{x+1}{x-3}$ as $x=\frac{3y+1}{y-1}$ introduces the notion of horizontal sign lines. In this form, we see that, with respect to x=0, $y=\frac{-1}{3}$ is a horizontal sign line and y=1 is a horizontal sign line as well as a horizontal asymptote.

[6] Our initial restriction on y=f(x) precludes such cases as y= x and y=x . These do satisfy the condition stated in this definition but do not have x=0 as a tangent carrier.

[7] Henry Frandsen, "The Last Word on Solving Inequalities," The Mathematics Teacher, LXII (October, 1969), 439-441.

8
ibid, p. 439
9
ibid, p. 441
10
Howard F. Fehr, *Secondary Mathematics: A Functional Approach for Teachers*, (D.C. Heath and Company, 1951), Chapter 5.
11
ibid, p. 88
12
ibid, p. 93

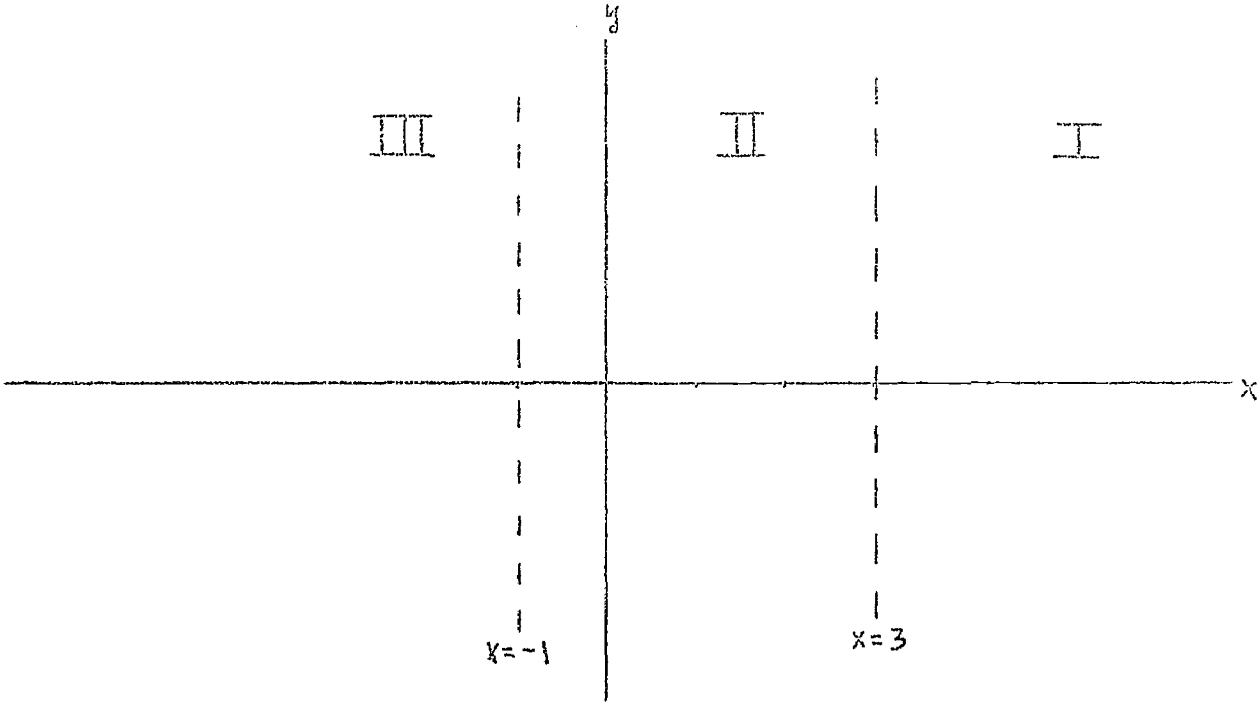

FIGURE 1

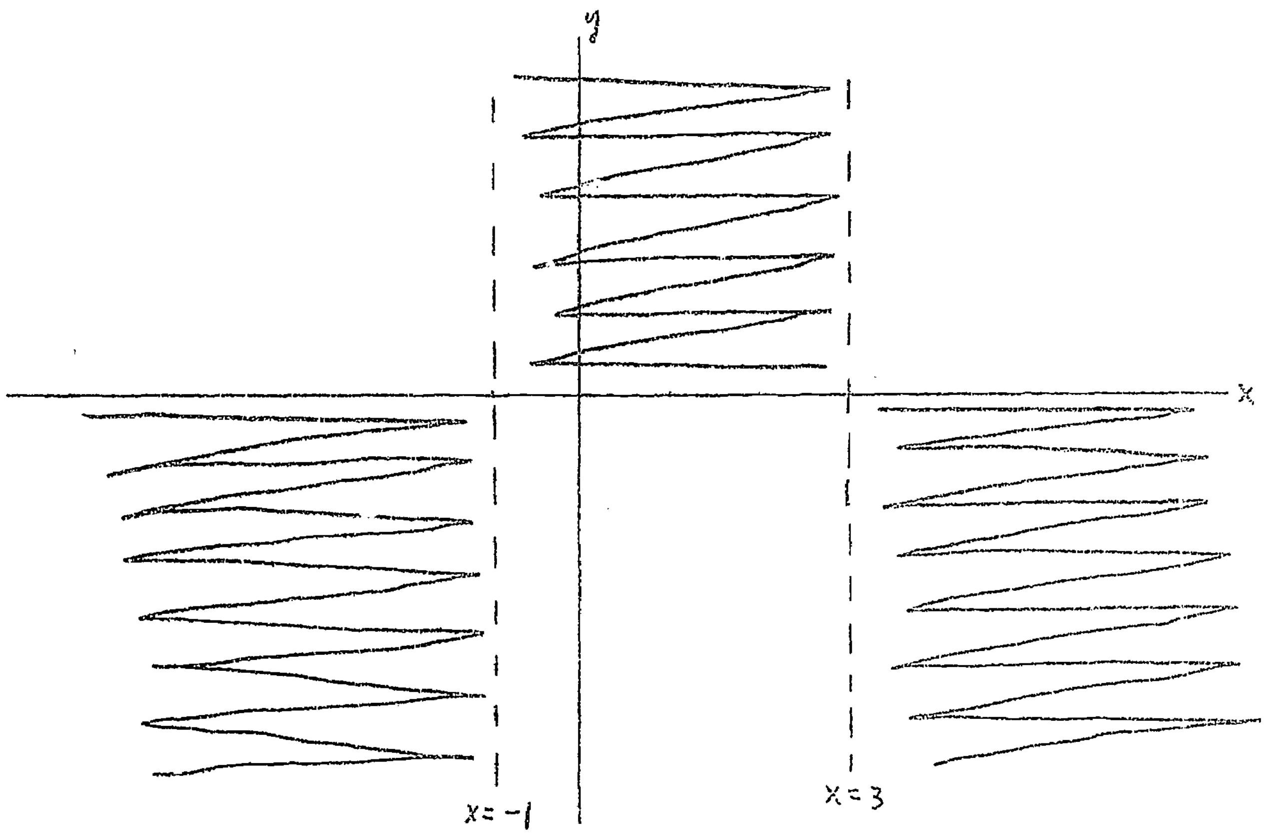

FIGURE 2

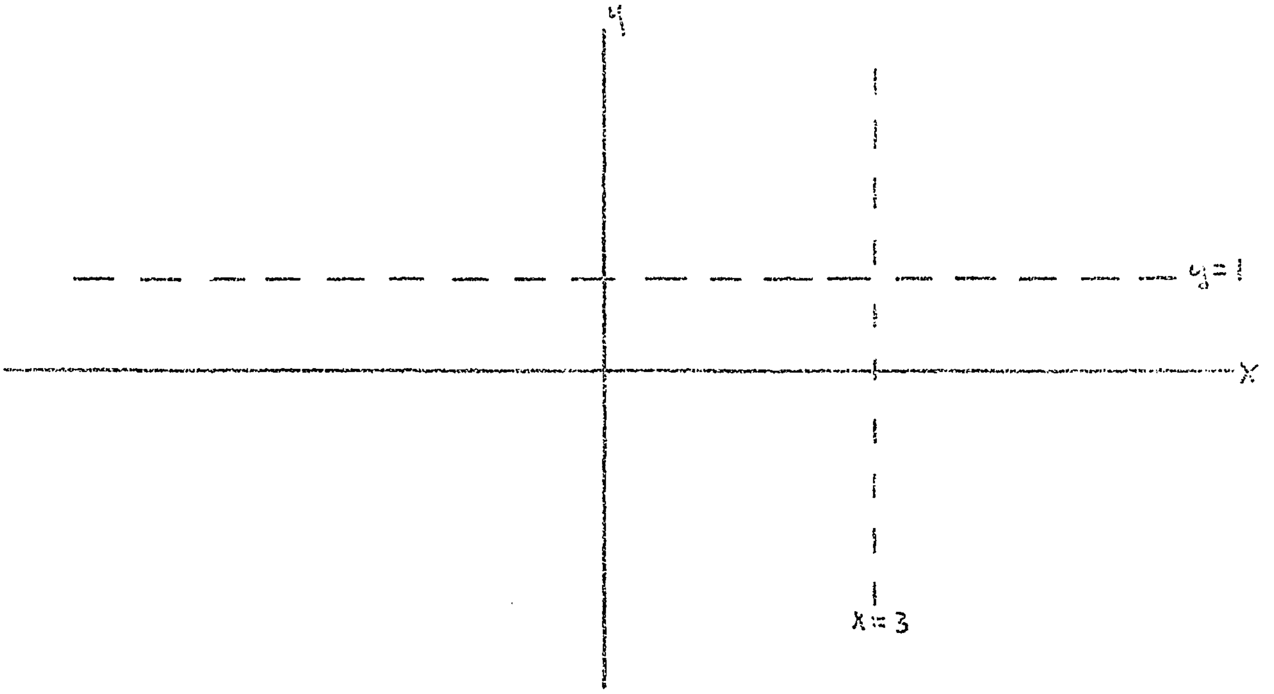

FIGURE 3

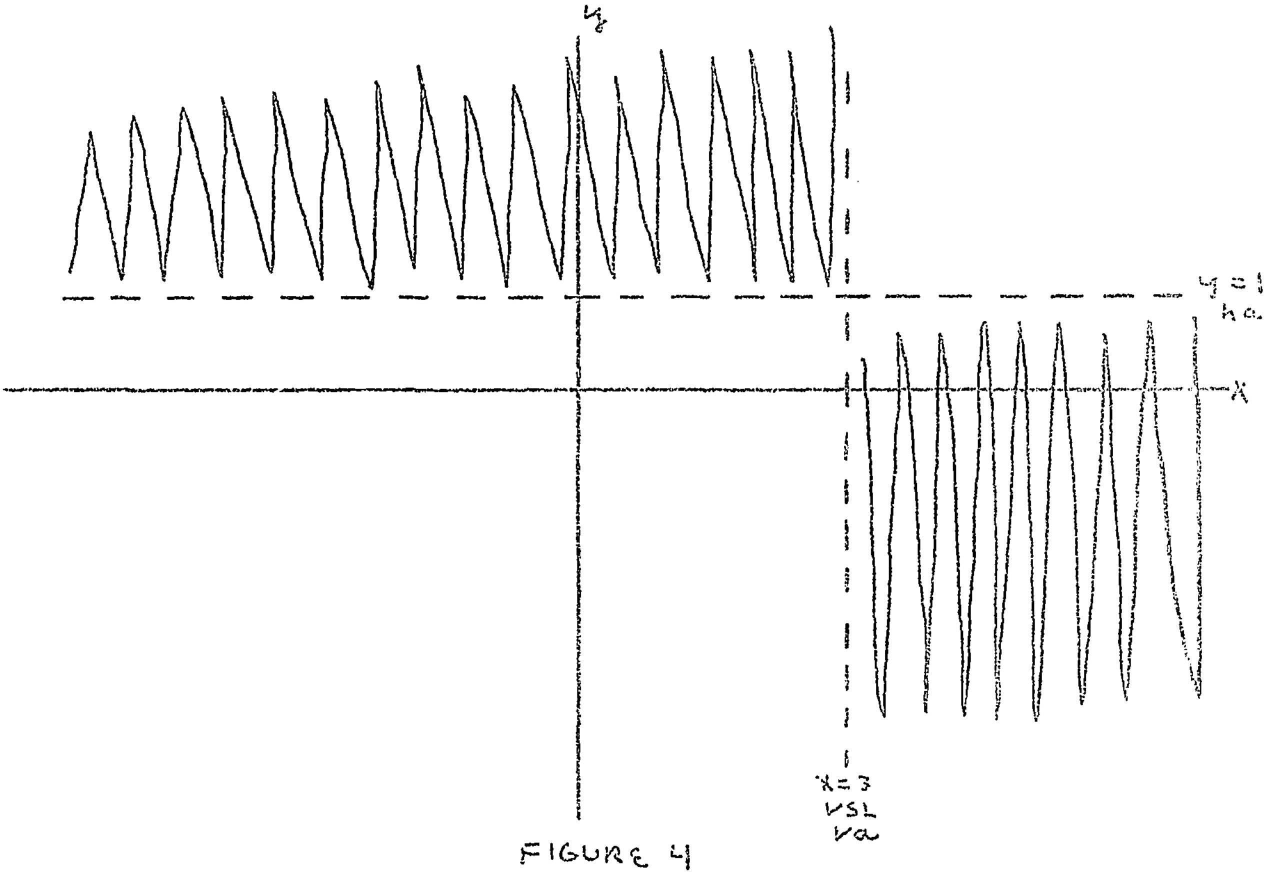

FIGURE 4

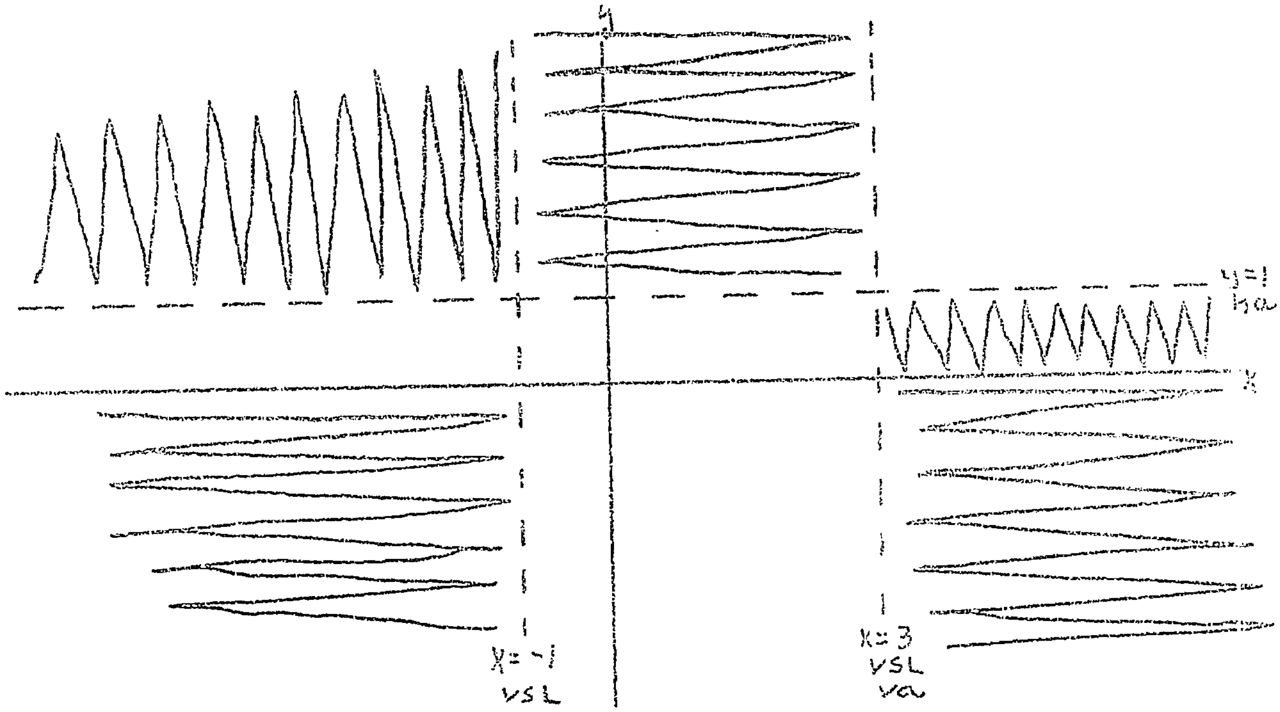

FIGURE 5

$$y = \frac{x+1}{x-3} = 1 + \frac{4}{x-3}$$

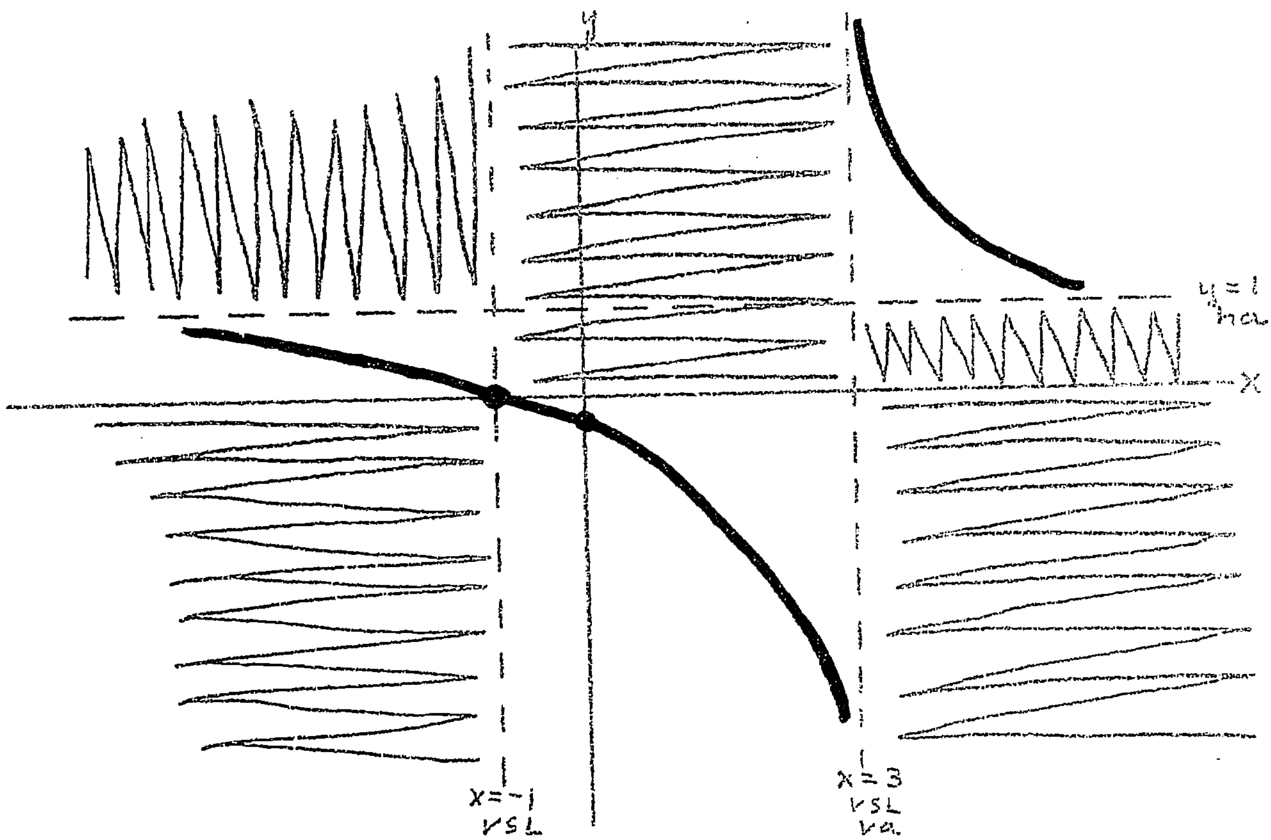

FIGURE 6

$$y = \frac{(x+1)(x-1)}{(x+2)(x-3)} = 1 + \frac{(x+5)}{(x+2)(x-3)}$$

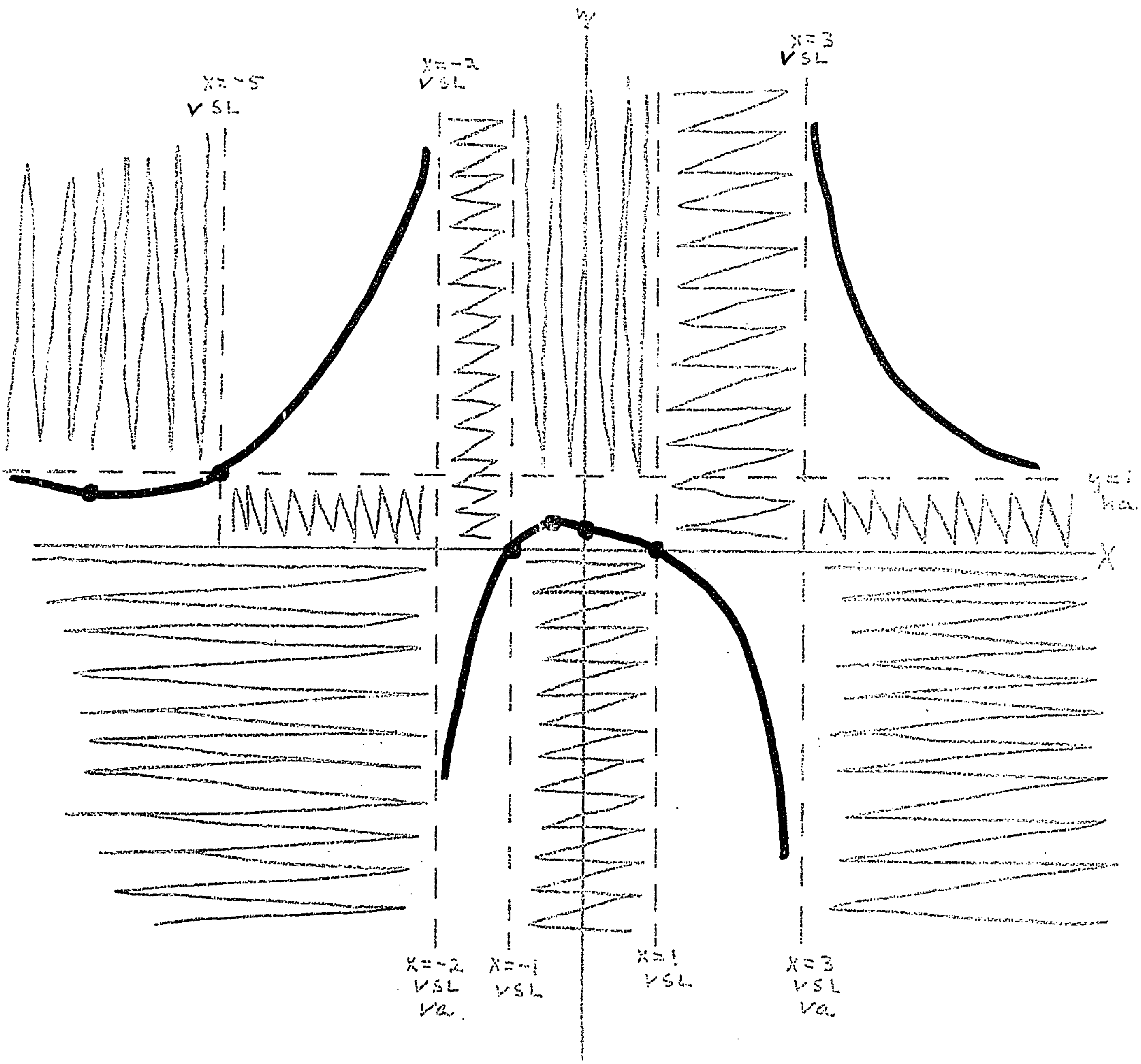

FIGURE 7

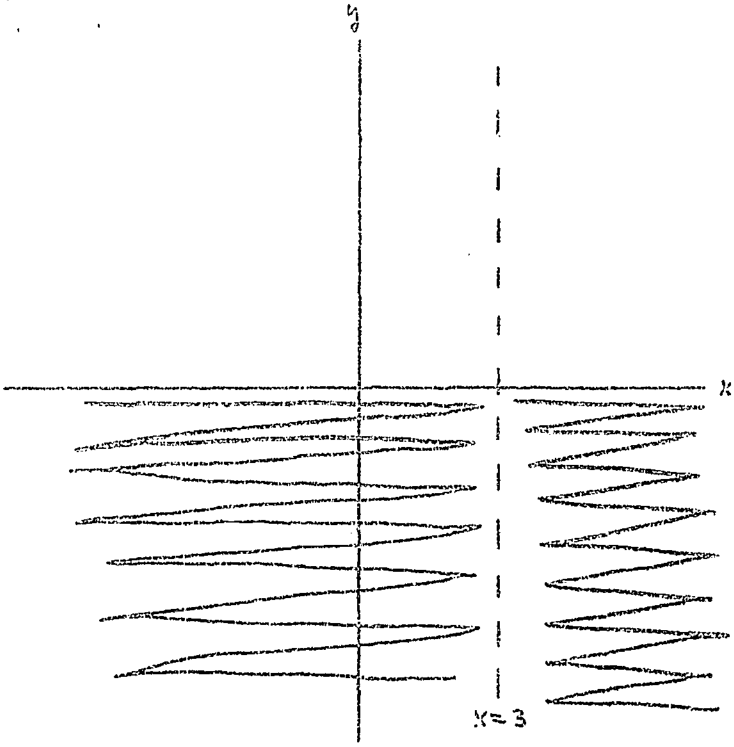

FIGURE 8

$y = (x-3)^2$

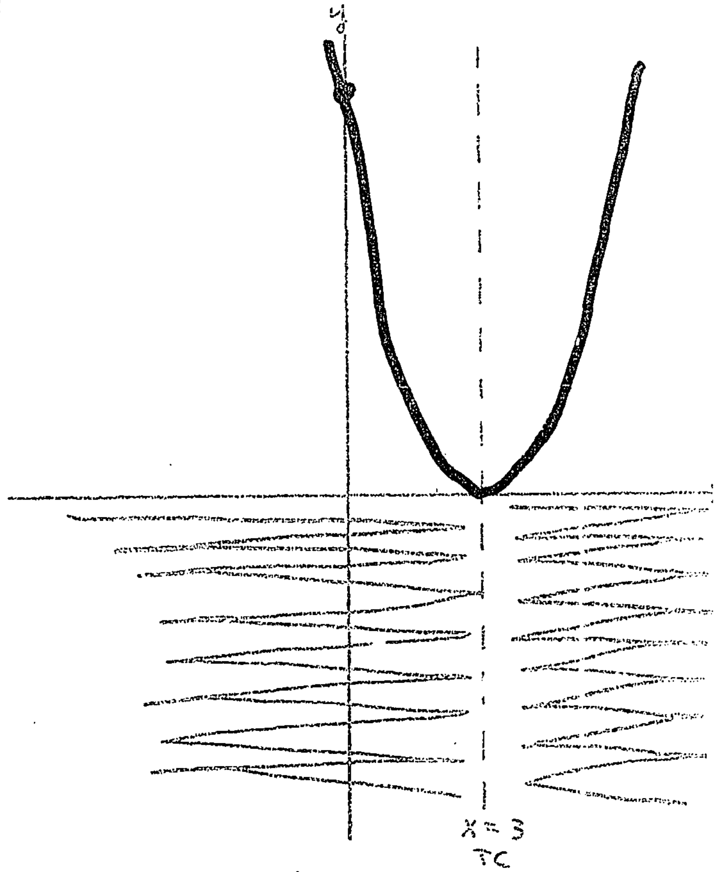

FIGURE 9

$y = \frac{1}{(x-3)^2}$

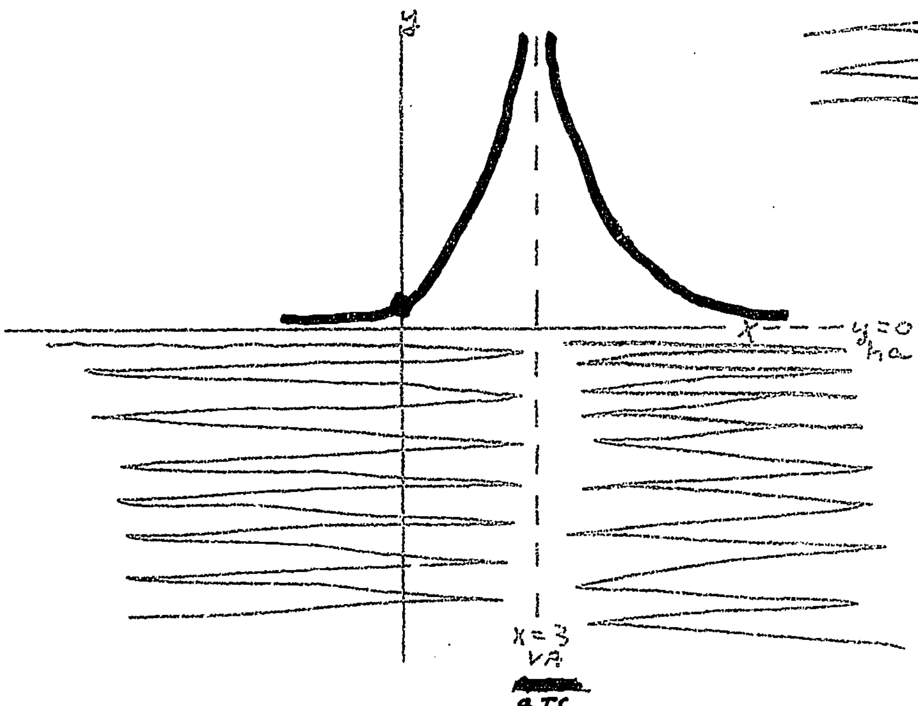

FIGURE 10

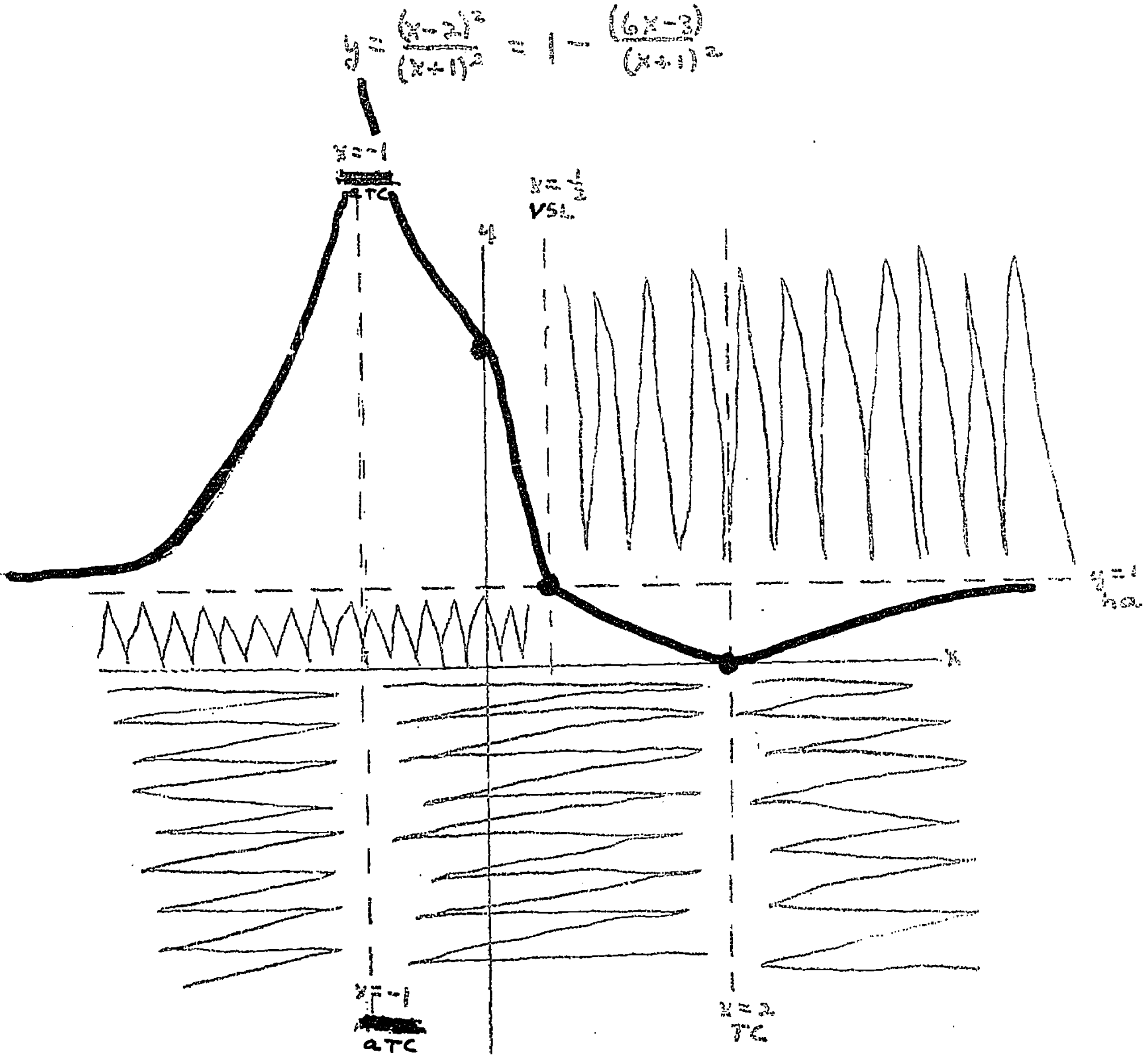
$y = \frac{(x-2)^2}{(x+1)^2} = 1 - \frac{(6x-3)}{(x+1)^2}$
$x = -1$
aTC
$x = \frac{1}{2}$
VSL
y
$y = 1$
hoa
x
$x = -1$
aTC
$x = 2$
TC

FIGURE 11

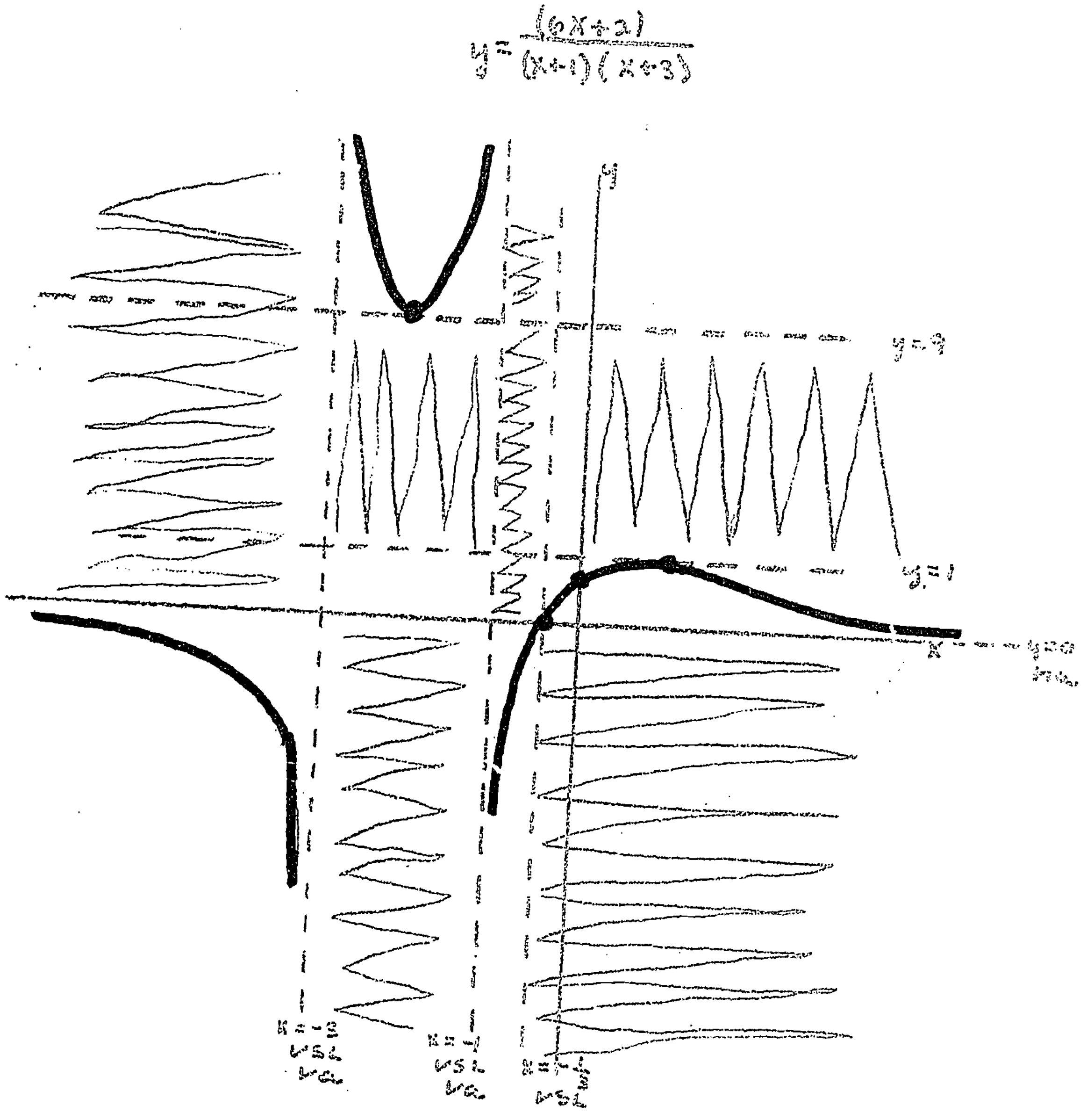

FIGURE 12

$$y = \frac{(x+4)^2}{(x+3)} = x + 5 + \frac{1}{(x+3)}$$

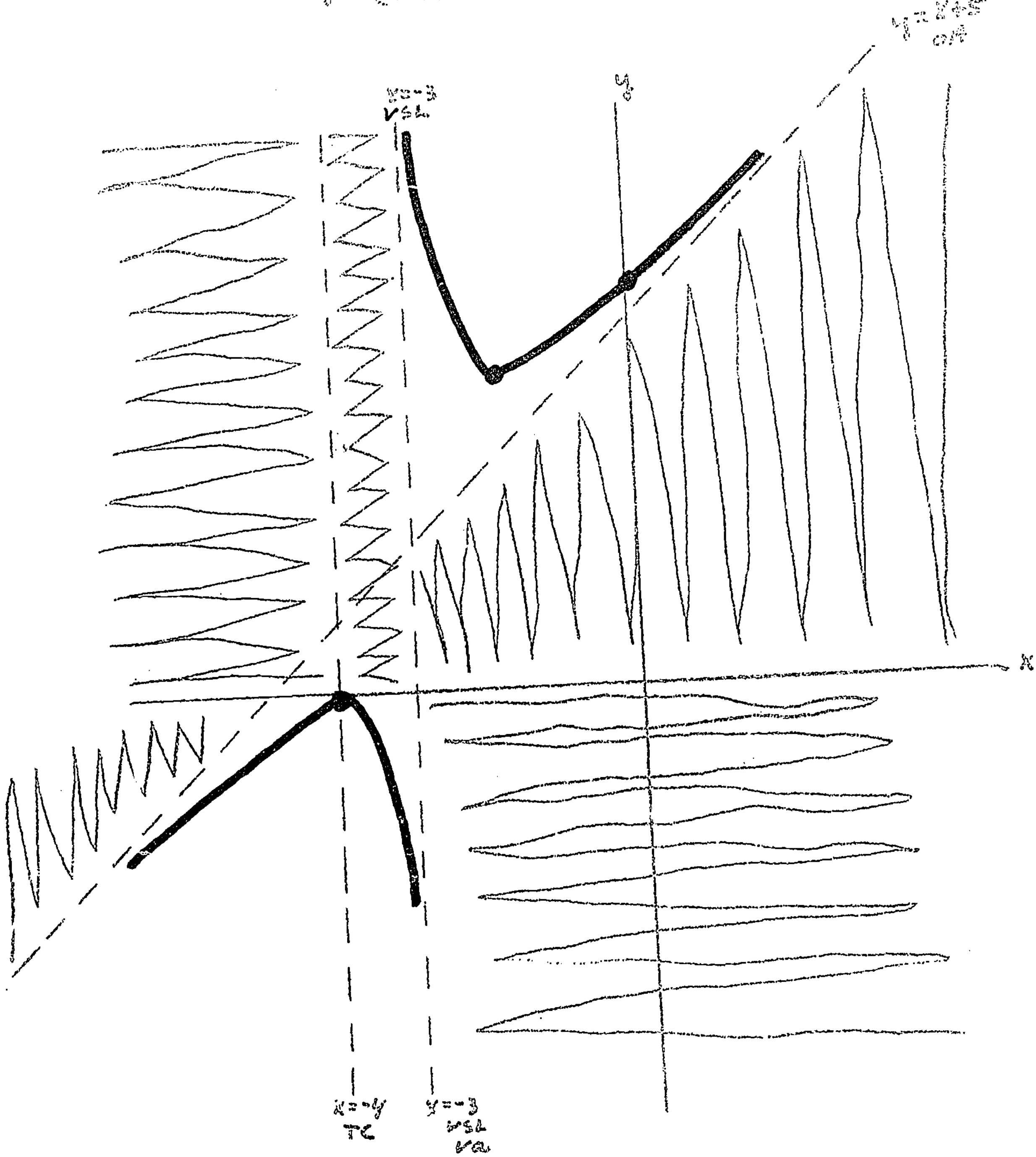

FIGURE 13

$$y = \frac{x^3 - 8}{x} = x^2 - \frac{8}{x}$$

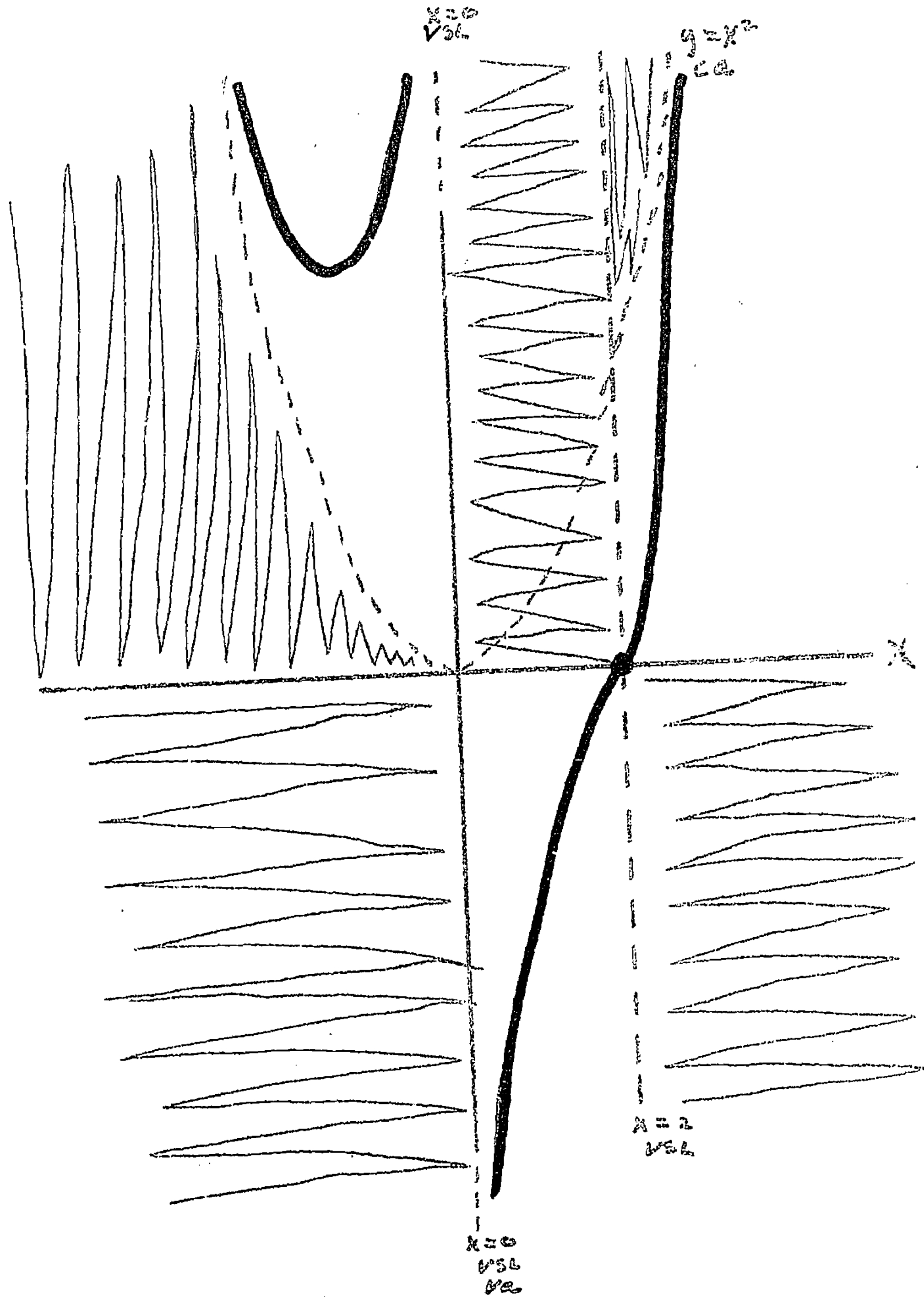

FIGURE 14

$$y = \frac{x^2}{(x+2)(x-1)} = 1 - \frac{x-2}{(x+2)(x-1)}$$

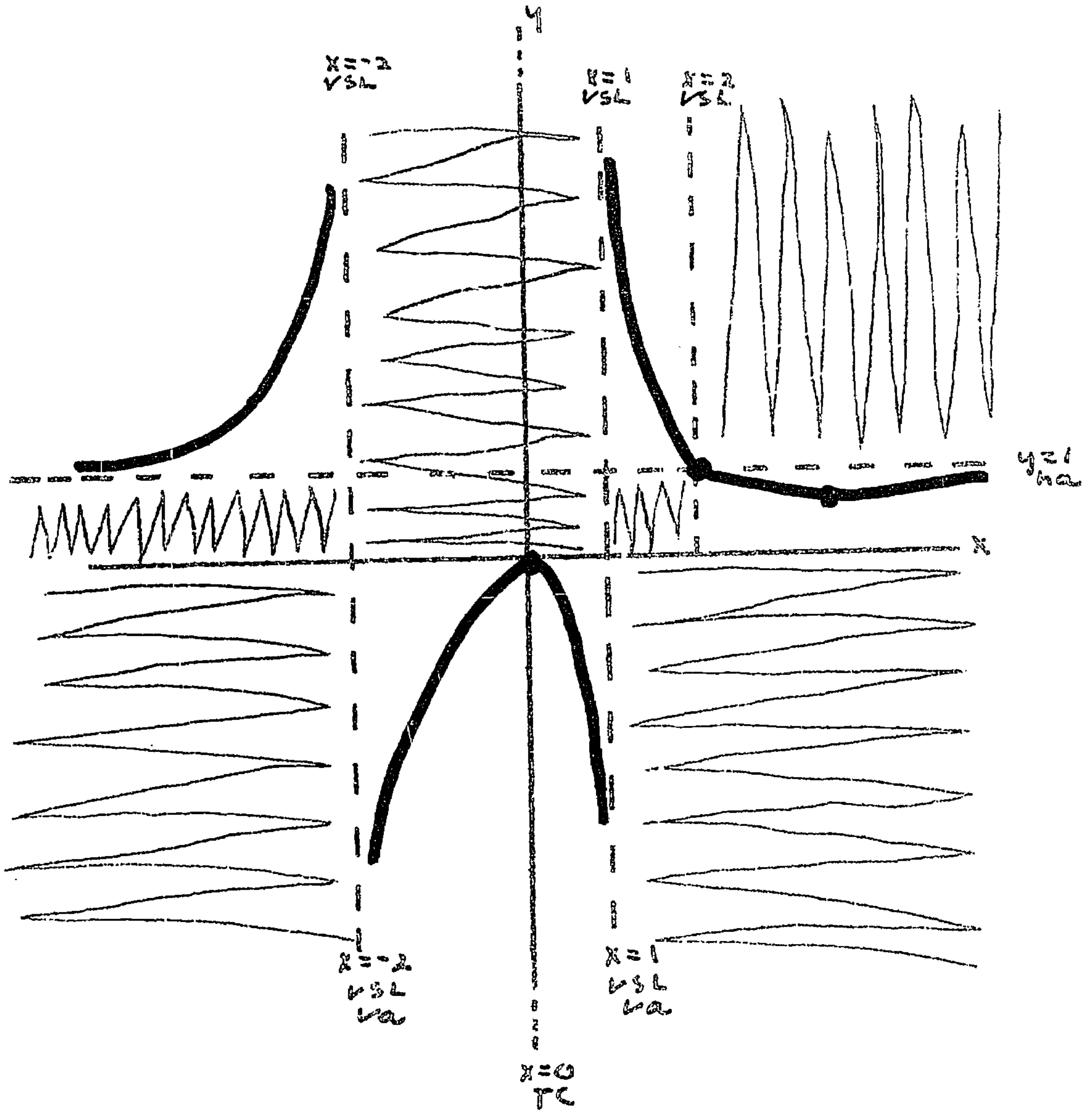

FIGURE 15

$$y = \frac{(x+1)^2}{x^2+1} = 1 + \frac{2x}{x^2+1} = 2 - \frac{(x-1)^2}{x^2+1}$$

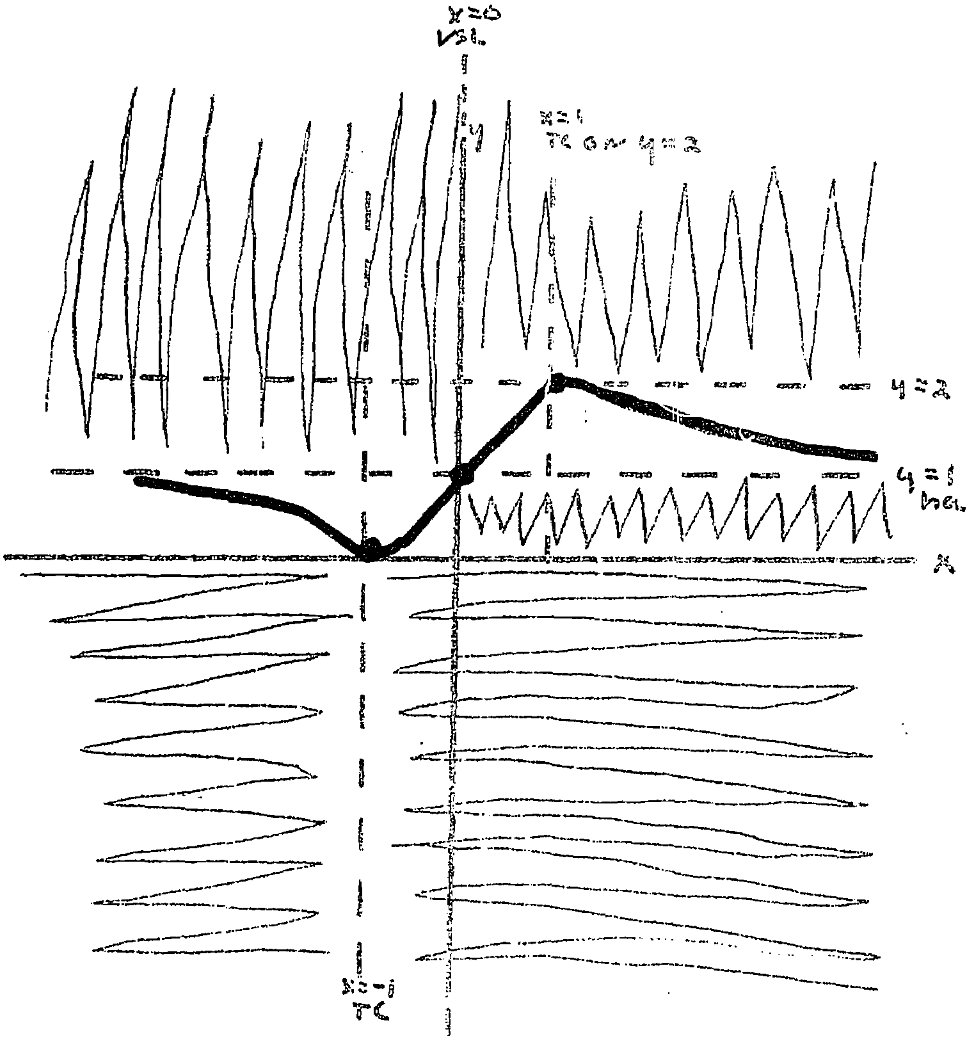

FIGURE 16

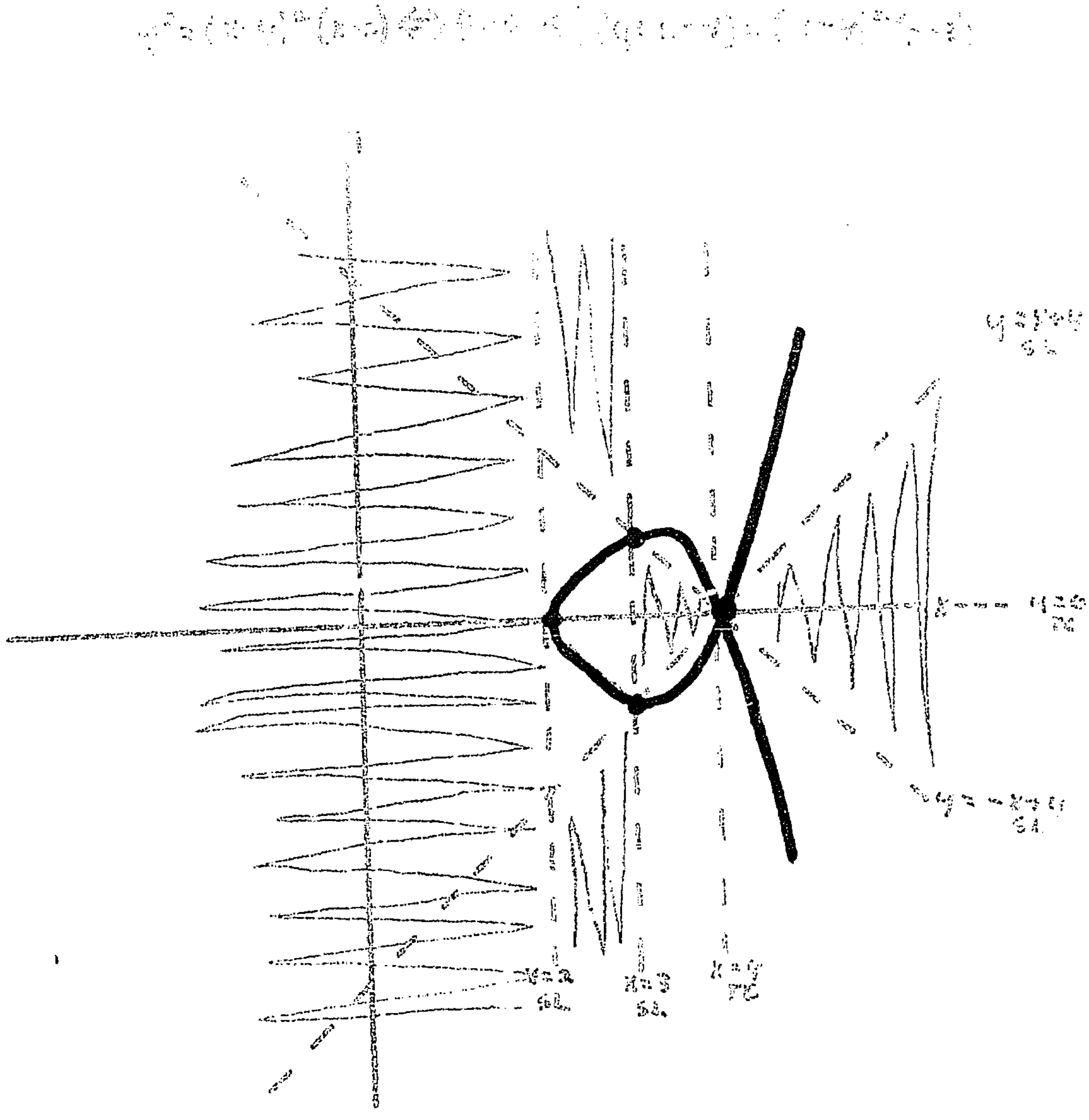

FIGURE 17

The Bow Tie

$$y^2 = x^2 - x^4 \iff y^2 = x^2(1-x)(1+x)$$
$$\iff x^4 = (x-y)(x+y)$$
$$\iff \left(x^2 - \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2} - y\right)\left(\tfrac{1}{2} + y\right)$$

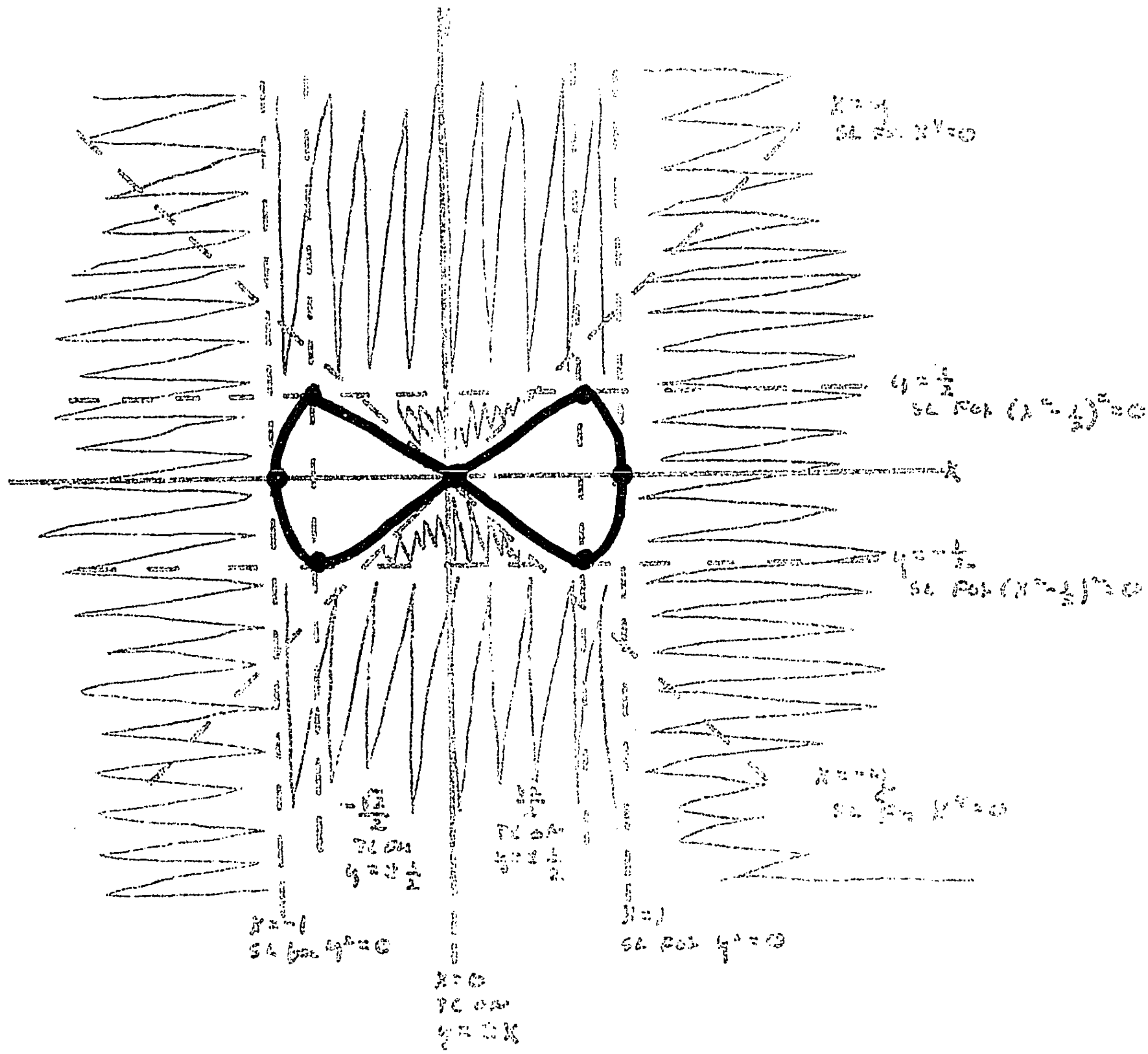

FIGURE 18

$$y = \frac{(x-1)(x+2)(x+3)}{(2x+1)(x)(x-2)}$$

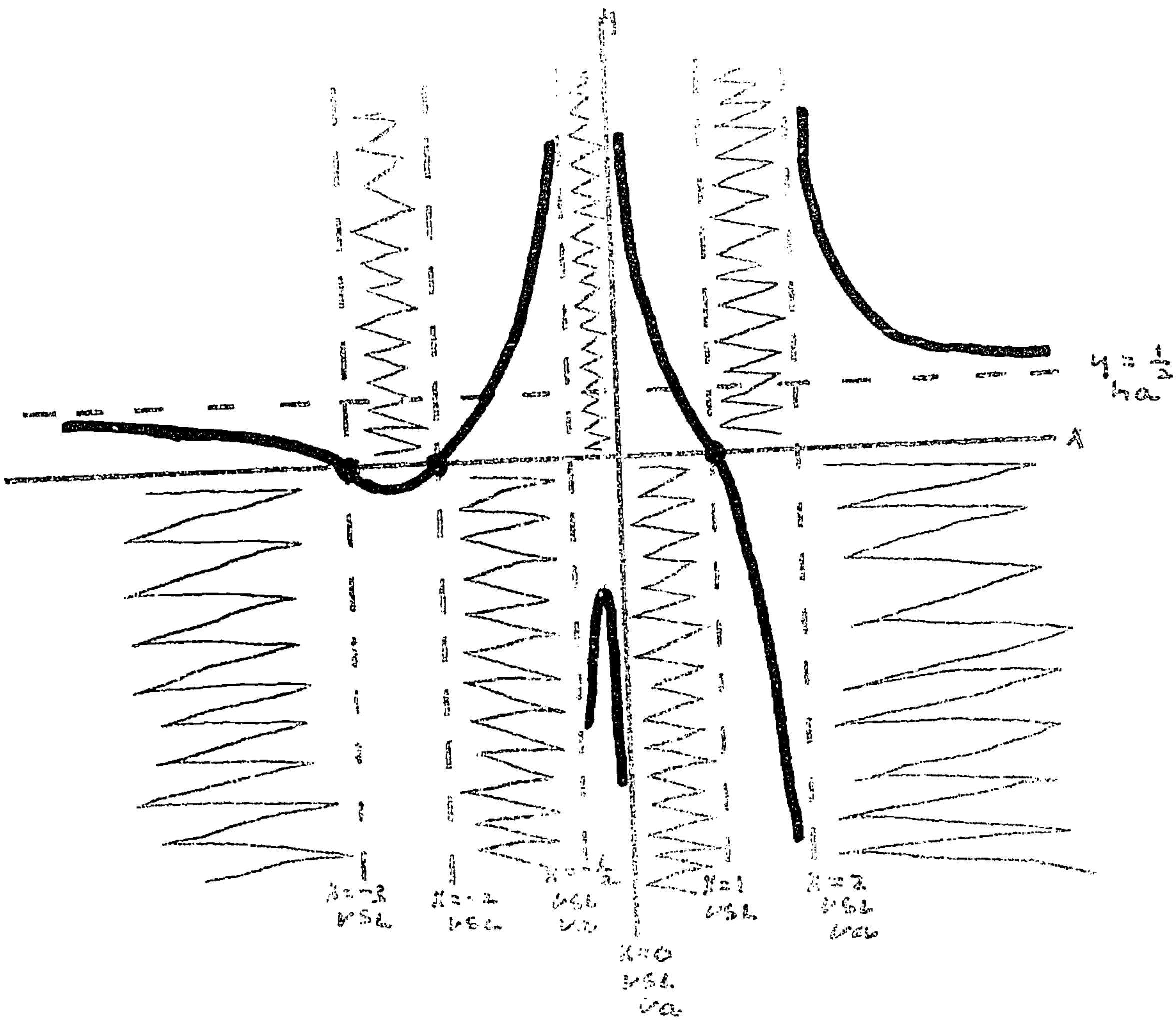

FIGURE 19